

96

EXPONENTIAL APPROXIMATIONS IN OPTIMAL DESIGN

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Abstract

One-point and two-point exponential functions have been developed and proved to be very effective approximations of structural response. The exponential has been compared to the linear, reciprocal and quadratic fit methods. Four test problems in structural analysis have been selected. The use of such approximations is attractive in structural optimization to reduce the numbers of exact analyses which involve computationally expensive finite element analysis.

1. INTRODUCTION

The use of detailed, computationally expensive, finite element models has motivated researchers to develop approximations of structural response. These approximations are useful for re-design particularly with use of optimization techniques, where the number of finite element analyses can be significantly reduced. The problem considered here is to construct local approximations using function values and derivatives of the structural response at one or two design points. The term local approximation used here means that the approximation is valid only in the vicinity of the current design point and is different from global approximation methods based on simplified design models or reduced basis techniques which seek to approximate the response in the entire design space.

Specifically, let \underline{x} be the current design point, where $\underline{x} = (x_1, x_2, \dots, x_n)^T$ is a design variable vector. Let $g(\underline{x})$ be a structural response such as element stress or fundamental frequency, which enters as a constraint function in an optimal design formulation. The problem is to construct a local approximation, $g_a(\underline{x})$, based on the function value and derivatives evaluated at \underline{x}^0 and possibly another design point. Then, subsequent evaluations of the structural response in the neighborhood of \underline{x}^0 can be estimated using g_a rather than the exact response g which will involve finite element computations. A variation of this problem is as follows: Let \underline{p} be a direction vector in design space which has been determined to be desirable in terms of reducing the cost function subject to constraints. Usually, \underline{p} is determined by solving a linear program or quadratic program in optimization algorithms. Now, let \underline{x}^1 be a second design point along \underline{p} such that $\|\underline{x}^1 - \underline{x}^0\|$ represents a move limit along \underline{p} . The problem is now to develop a (local) line approximation $g_a(\underline{x})$ such that $g_a(\underline{x}) \approx g(\underline{x})$ for points \underline{x} along the line joining \underline{x}^0 and \underline{x}^1 , given by

$$\underline{x} = (1-\zeta) \underline{x}^0 + \zeta \underline{x}^1, \quad 0 \leq \zeta \leq 1 \quad (1)$$

Here, the approximation g_a is to be constructed using structural response information at \underline{x}^0 and possibly \underline{x}^1 .

A comparison of various approximation methods has been carried out by Haftka, *et. al.* (ref. 1) and Haftka (ref. 2). The methods include linear and quadratic Taylor series expansions involving first order and second order sensitivity analysis (refs. 3-5), approximations based on use of reciprocal design variables (refs. 6, 7) and convex approximations (ref. 8). Recently, force approximations have been used by Vanderplaats (ref. 9). Use of rational polynomials may be found in Ref. 10. In this paper, exponential approximations of the form

$$g_a = C \prod_{i=1}^n x_i^{a_i} \quad (2)$$

are considered and compared with linear, reciprocal and quadratic polynomial methods. It is noted the exponential approximation discussed in Ref. (1) is of a different form than that in (2). The motivation for choosing exponential approximations of the form in (2) is discussed below.

2. BASIS FOR EXPONENTIAL APPROXIMATIONS

The motivation for approximating structural response using the exponential form in (2) is discussed in this section, as also the basis for use of reciprocal variables and force approximations. The basis for most approximations comes from the equation

$$\sigma(A) = \frac{P}{A} \quad (3)$$

which states that stress = element force/area. Area A is the design variable here.

Reciprocal Variables

The choice of reciprocal design variables is natural, since choosing $x = 1/A$ as a variable results in $\hat{\sigma} = \sigma(1/x)$ being linear in x :

$$\hat{\sigma}(x) = P x \quad (4)$$

In the x -space, larger more limits can be imposed on changes in design, leading to faster convergence. Now, in a statically indeterminate truss, the stress function is of the form

$$\sigma(A) = \frac{P(A)}{A} \quad (5)$$

The force P is no longer a constant, but dependent on design. The choice of $x=1/A$ is still beneficial as it tends to linearize the stress function. In general, a first order Taylor service expansion of $g(\underline{x})$ in the reciprocals of the variables $y_i, = 1/x_i, i=1, \dots, n$, written in terms of the original variables, x_i , is given by

$$g_a(\underline{x}) = g(\underline{x}^0) + \sum_{i=1}^n (x_i - x_{i0}) \left(\frac{x_{i0}}{x_i} \right) \partial g / \partial x_i \quad (6)$$

Force Approximations

The idea here is to approximate $P(A)$ in (5) by Taylor series as opposed to $\sigma(A)$, and obtain

$$\sigma_a(A) = \frac{P(A_0) + \frac{dP}{dA}(A_0) (A - A_0)}{A} \quad (7)$$

In the case when P is a constant, the approximation yields $\sigma_a = P/A$ which is exact. Otherwise, curvature information is retained in (7) and yields a superior approximation to the conventional tangent approximation $\sigma_a = \sigma(A_0) + \partial\sigma/\partial A \cdot (A-A_0)$.

Exponential Approximations

The approximation introduced in this paper is now discussed. Equation (3) may be re-written as

$$\sigma(A) = P A^{-1} \quad (8)$$

Thus, the stress is seen to be exponentially related to design variable A. This is the basis for approximating structural response in n-dimensional space by

$$g_a(\underline{x}) = C \prod_{i=1}^n x_i^{a_i}$$

where x_1, x_2, \dots, x_n are non-negative design variables. Choice of constants C and a_i are discussed in the next section.

A second and more general basis for exponential approximations lies in the concept of 'elasticity', a quantity used by economists and also relevant in nonlinear stress-strain constitutive laws. Consider a function $g = g(x)$, where $x > 0$ is a scalar variable. The elasticity of the function is defined as

$$e_g = \frac{d(\ln g)}{d(\ln x)} \quad (9a)$$

or,

$$e_g = \frac{dg/g}{dx/x} \quad (9b)$$

Physically, elasticity may be considered to be in the limit, the percentage change in the function due to a percentage change in the variable. For instance, $g=x^3$ has a value $e_g=3$, and $g=px^{-1}$ has $e_g = -1$. The exponents a_i in (2) may be considered to be estimates of the elasticity at the current design point.

In this section, reciprocal and force approximation methods have been introduced using the fundamental equation $\sigma = P/A$ as a basis. Work is being done to generalize these methods to be applicable to frames and certain elasticity problems as well. The exponential method of approximation has both $\sigma=P/A$ as a basis as well as the concept of elasticity of a function. One advantage of exponential approximations of the form in (2) is that, for $C > 0$, the function g_a is a monomial, which opens up the possibility of geometric programming (ref. 11).

3. CONSTRUCTION OF THE EXPONENTIAL APPROXIMATION

The problem is to find the coefficient C and exponents $a_i, i=1, \dots, k$, such that the approximate function $g_a(\underline{x}) = C \prod_{i=1}^n x_i^{a_i}$ closely matches the exact function $g(\underline{x})$ in a neighborhood of the current point \underline{x}^0 . One-point and two-point approximations will now be given.

1-Point Approximation

Here, constants C and $\{a_i\}$ are determined using information only at one point \underline{x}^0 . The technique is based on matching the function value and shapes of g_a and g . This technique has been used in the context of unconstrained geometric programming where general functions are reduced to posynomial form. Morris (ref. 11) discusses an application of this concept to structural design problems. We have, upon taking logarithms,

$$\ln g_a = \ln C + \sum_{i=1}^n a_i \ln x_i \quad (10)$$

Differentiating with respect to x_j yields

$$\partial g_a / \partial x_j = g_a \left[\frac{a_j}{x_j} \right] \quad (11)$$

Note that $g_a(\underline{x}^0) = g(\underline{x}^0)$. Equating $\partial g_a / \partial x_j$ in (11) to the exact slope $\partial g / \partial x_j$ at \underline{x}^0 yields the exponents

$$a_j = \frac{x_j \partial g / \partial x_j}{g} \Big|_{\underline{x}^0} \quad (12)$$

The coefficient C is then obtained from $g_a(\underline{x}^0) = g(\underline{x}^0)$ as

$$C = g / \prod_i x_i^{a_i} \Big|_{\underline{x}^0} \quad (13)$$

2-Point Approximation

Information at two points are used to construct the exponential approximation. Let \underline{x}^0 be the current design point and \underline{x}^1 be a second point, which usually is a point along a desired search direction in design space. The quantities $g(\underline{x}^0)$, $\nabla g(\underline{x}^0)$ and $g(\underline{x}^1)$ are now used to determine C and $\{a_i\}$. A least squares formulation is adopted herein. The variable C and $\{a_i\}$ are obtained from the minimization problem

$$E = \frac{1}{2} \left\{ \left(g_0 - C \prod_{i=1}^n x_{i0}^{a_i} \right)^2 + \left(g_1 - C \prod_{i=1}^n x_{i1}^{a_i} \right)^2 + \sum_{j=1}^n \left(\frac{\partial g_0}{\partial x_j} - C \prod_{i=1}^n x_{i0}^{a_i} \cdot \ln x_j \right)^2 \right\} \quad (14)$$

The minimization of the least squares objective function E is carried out using a modified Newton algorithm, with a Levenberg-Marquardt correction to the Hessian when descent is not obtained (ref. 12). The algorithm requires the gradient vector

$$\nabla E = (\partial E / \partial C, \partial E / \partial a_1, \dots, \partial E / \partial a_n) \quad (15)$$

and the Hessian

$$H_E = \begin{bmatrix} \partial^2 E / \partial C^2 & \partial^2 E / \partial C \partial a_1 \dots & \partial^2 E / \partial C \partial a_n \\ & \partial^2 E / \partial a_1^2 \dots & \partial^2 E / \partial a_1 \partial a_n \\ \text{symmetric} & & \ddots & \partial^2 E / \partial a_n^2 \end{bmatrix} \quad (16)$$

These derivatives are computed from analytically derived expressions. The least squares algorithm is given below.

Algorithm 2-Point Exponential

Step 1. Choose the initial estimates of C and $\{a_i\}$ from (12), (13), and $\epsilon_0 = 0.001$

Step 2. Solve

$$(\underline{H}_E + \epsilon \underline{I}) \underline{\delta} = -\nabla E^T \quad (17)$$

and update $C_{\text{new}} = C + \delta_1$, $(a_i)_{\text{new}} = a_i + \delta_{i+1}$, $i = 1, \dots, n$.

Step 3. Evaluate E_{new} . If $E_{\text{new}} < E$ the set $C = C_{\text{new}}$, $\{a_i\} = \{a_i\}_{\text{new}}$, reduce ϵ , say, $\epsilon = \epsilon/10$ (if $\epsilon < \epsilon_0$, set $\epsilon = \epsilon_0$) and go to step 2. If $E_{\text{new}} > E$, then increase $\epsilon = 10\epsilon$ and go to step 2.

The procedure above is terminated when relative and absolute reductions in E for three consecutive iterations are less than a specified tolerance.

4. TEST PROBLEMS AND RESULTS

Four test problems relating to structural design have been considered. The 1-point and 2-point exponential approximations developed in Section 3 are examined. Comparison of the approximation to the original function is done along a line joining two design points \underline{x}^0 , \underline{x}^1 , or at points \underline{x} where

$$\underline{x} = (1 - \zeta) \underline{x}^0 + \zeta \underline{x}^1 \quad (18)$$

where ζ is scalar variable, $0 \leq \zeta \leq 1$. For comparison, the linear (tangent) approximation based on

$$g_a(\underline{x}) = g(\underline{x}^0) + \nabla g(\underline{x}^0) \bullet (\underline{x} - \underline{x}^0) \quad (19)$$

the reciprocal-linear approximation given in (6), and the quadratic polynomial along the line given by

$$g_a(\zeta) = a + b\zeta + c\zeta^2 \quad (20)$$

where coefficients a , b , c are obtained from $g(\underline{x}^0)$, $g(\underline{x}^1)$ and $dg/d\zeta$ (at $\zeta=0$) = $\nabla g(\underline{x}^0) \bullet (\underline{x}^1 - \underline{x}^0)$. Thus, the 1-point exponential, linear, and reciprocal require only $g(\underline{x}^0)$, $\nabla g(\underline{x}^0)$, while the 2-point exponential and quadratic polynomial require, in addition, $g(\underline{x}^1)$. The error between g and g_a along the line is shown both graphically as well as quantitatively through a relative error criterion

$$\text{RELEER} = \sqrt{\sum_{i=1}^N [(g_i - g_{ai}) / g_i]^2} \quad (21)$$

and a maximum error criterion

$$\text{MAXERR} = \max_{1 \leq i \leq N} |g_i - g_{ai}| \quad (22)$$

Above, $g_i = g(x(\zeta_i))$ is the exact function evaluated at the i^{th} discretization point along the line in (18), g_{ai} is the approximate function evaluated at ζ_i , and N , the number of discretization points, is chosen equal to 20.

Cantilever Beam

The axial stress function in a cantilever beam of rectangular cross section, subjected to axial and transverse loads, is given as

$$\sigma(\underline{x}) = \frac{1000}{x_1 x_2} + \frac{6000}{x_1 x_2^2} \quad (23)$$

where x_1, x_2 are the width and depth of the cross section, respectively. The choice of design points is $\underline{x}^0 = (1, 2)^T$ in., $\underline{x}^1 = (5, 8)^T$ in.

Referring to Fig. 1, the 2-point exponential is in excellent agreement with the original function. The 1-point exponential behaves just as well as the 2-point exponential and is not shown in the figure. The exponential approximation to $\sigma(\underline{x})$ in (23) is of the form

$$\sigma_a(\underline{x}) = 6727.2 x_1^{-0.891} x_2^{-1.750} \quad (24)$$

The quadratic polynomial (Fig. 1), as well as the tangent and reciprocal approximations behave very poorly. The values of RELEER and MAXERR in (21), (22) for this problem are given in Table 1. It is noted that various choices of \underline{x}^0 and \underline{x}^1 have shown the same trend.

Table 1. Cantilever Beam

Approximation Method	Relative Error	Maximum Error
1-point exponential	0.493	13.1
2-point exponential	0.492	13.0
Linear (tangent)	700.0	0.165 E5
Reciprocal	105.4	0.022 E5
Quadratic Polynomial	96.9	0.033 E5

Tension-Compression Spring

The shear stress function in a spring design problem, with x_1 = coil diameter and x_2 = wire diameter, is given by

$$\tau(\underline{x}) = \frac{8000 x_1}{\pi x_2^3} \left[\left(\frac{4 x_1 - x_2}{4 x_1 - 4 x_2} \right) + 0.615 \frac{x_2}{x_1} \right] \quad (25)$$

For this problem, $\underline{x}^0 = (1.0, 0.3)^T$ and $\underline{x}^1 = (0.3, 0.05)^T$ in. As with the beam, the 2-point and 1-point exponentials are also in close agreement with the original function. Table 2 provides an error summary for all the methods. The linear, reciprocal and quadratic polynomial are poor by comparison (Fig. 2). Other choices of $\underline{x}^0, \underline{x}^1$ show the same trend for this problem.

Table 2. Tension-Compression Spring

Approximation Method	Relative Error	Maximum Error ($\times 10^6$)
1-point exponential	0.091	0.496
2-point exponential	0.088	0.067
Linear (tangent)	2.464	7.265
Reciprocal	1.640	5.876
Quadratic Polynomial	11.060	3.539

Three Bar Symmetrical Truss

The natural frequency of a three bar truss (ref. 3) with x_1, x_2 = cross sectional areas, is described by the function

$$\omega(\underline{x}) = \frac{x_1}{2\sqrt{2} x_1 + x_2} \quad (26)$$

Two sets of design points, leading to different performances, are chosen. These sets are

$$\begin{aligned} \text{I.} \quad & \underline{x}^0 = (3, 4)^T \text{ in}^2, \quad \underline{x}^1 = (10, 5)^T \text{ in}^2 \\ \text{II.} \quad & \underline{x}^0 = (5, 5)^T \text{ in}^2, \quad \underline{x}^1 = (1, 10)^T \text{ in}^2 \end{aligned} \quad (27)$$

Set I: Referring to Fig. 3 and Table 3, the 2-point exponential based on the best fit formulation yields the best approximation, with

$$\omega(\underline{x}) = 0.2877 x_1^{0.261} x_2^{-0.342} \quad (28)$$

The 1-point exponential is poorer, with

$$\omega(\underline{x}) = 0.2635 \ x_1^{0.32} \ x_2^{-0.32} \quad (29)$$

Set II: Along the search direction, the original function is quite flat. In fact, the 1-point exponential provides a relatively poor approximation because of the flat nature of the function. The quadratic polynomial is best here. Even though the 2-point exponential is second-best, (Fig. 4), the best-fit nature of the approximation, while averaging the error, does not provide an interval within the line where the error is small. This may cause difficulty for designs near the optimum. Finally, the use of reciprocal variables does not show any advantage over the direct variables for this case.

Table 3. Three Bar Symmetrical Truss

Approximation Method	(Set I)		(Set II)	
	Relative Error	Maximum Error	Relative Error	Maximum Error
1-point exponential	0.247	0.0286	1.386	0.065
2-point exponential	0.059	0.0053	0.605	0.028
Linear (tangent)	0.810	0.100	1.210	0.060
Reciprocal	0.199	0.022	1.902	0.060
Quadratic Polynomial	0.128	0.012	0.074	0.004

Ten Bar Truss

The ten cross sectional areas of the truss shown in Fig. 5 are the design variables. Points \underline{x}^0 and \underline{x}^1 are chosen as the initial and optimum design obtained in Ref. (1), as

$$\begin{aligned} \underline{x}^0 &= (5., 5., 5., 5., 5., 5., 5., 5., 5., 5.)^T \text{ in}^2 \\ \text{and} \quad \underline{x}^1 &= (7.94, 0.1, 8.06, 3.94, 0.1, 0.1, 5.74, 5.57, 5.57, 0.1)^T \text{ in}^2 \end{aligned} \quad (30)$$

Again, both 1-point and 2-point exponentials provide excellent approximations. The number of design variables do not seem to affect their quality. Interestingly, the reciprocal approximation provides equally good results, but to within a certain distance from \underline{x}^0 . Near \underline{x}^1 , the reciprocal abruptly diverges (Fig. 6). With smaller move limits, of course, the reciprocal will be excellent for this problem. The quadratic polynomial provides a good approximation for this problem. Error estimates are given in Table 4.

Table 4. Ten Bar Truss

Approximation Method	Relative Error	Maximum Error
1-point exponential	0.088	1.528
2-point exponential	0.021	0.204
Linear (tangent)	0.689	7.907
Reciprocal	0.599	13.980
Quadratic Polynomial	0.081	0.803

5. CONCLUSIONS

Exponential functions of the form $C \prod_i x_i^{a_i}$ have been used to approximate structural response. Both 1-point and 2-point approximations have been used to determine C and $\{a_i\}$. The 1-point involves matching function and derivative values at the current design. The 2-point method is based on minimizing a least squares function by modified Newton's method. The basis for exponential approximations is from two sources: one is from structural theory, where $\sigma = P/A$ can be written as $\sigma = PA^{-1}$, while the other is from economics, where a function $g = cx^a$ has an elasticity equal to the exponent a . The restriction of exponential approximations is $x_i > 0$. An advantage is that the approximating function is valid for any type of structure or type of structural response. Further, the exponential approximations when applied to the cost and constraints of an optimal design problem have the potential for being used in conjunction with geometric programming which can effectively solve the subproblem.

Results on three out of the four structural problems considered have shown that the exponential functions have provided excellent approximations, with essentially no error, even for large distances in the design space. The linear, linear-reciprocal and quadratic polynomial are much inferior.

On one of the problems involving natural frequency of a 3-bar truss, it is observed that the function is essentially flat or linear. In this case, the 2-point exponential (based on a best-fit) does better than the 1-point, but the quadratic polynomial is superior. Thus, for linear or nearly linear functions, the linear approximation is to be preferred. For other cases, the exponential has shown to be a powerful method of approximation.

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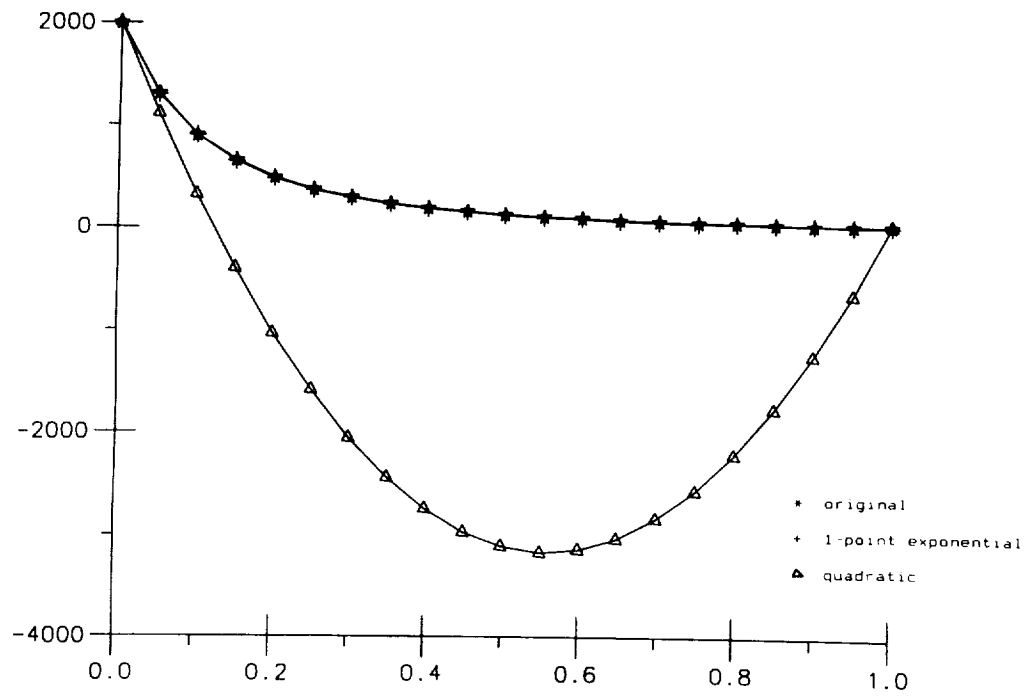


Figure 1. Approximations for the Cantilever Beam.

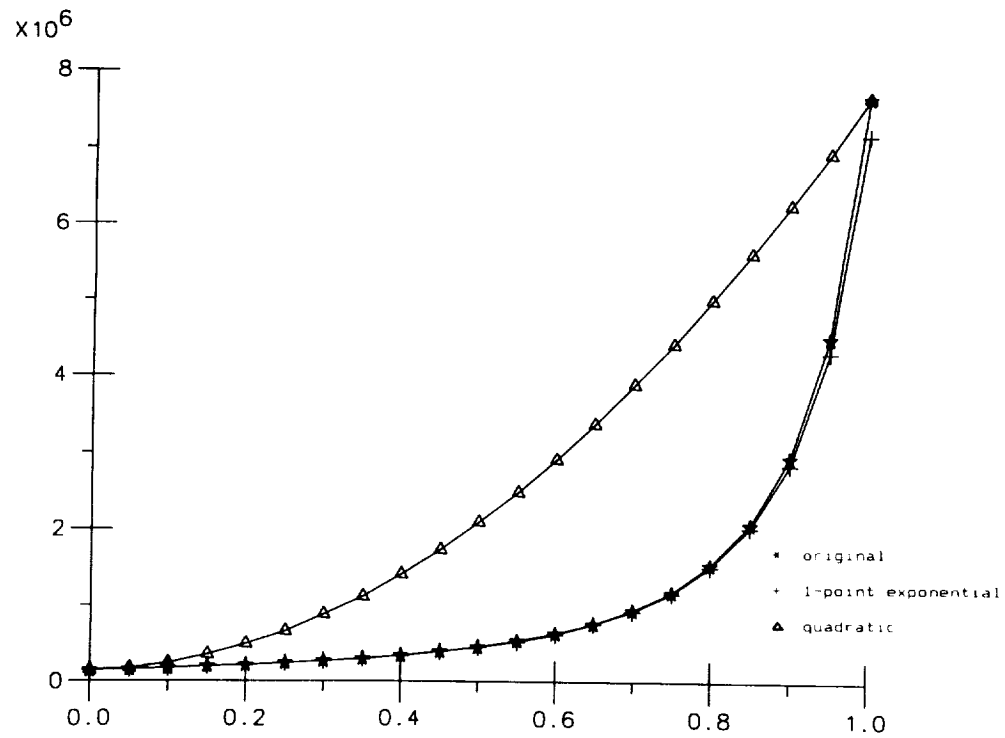


Figure 2. Approximations for the Tension-Compression Spring.

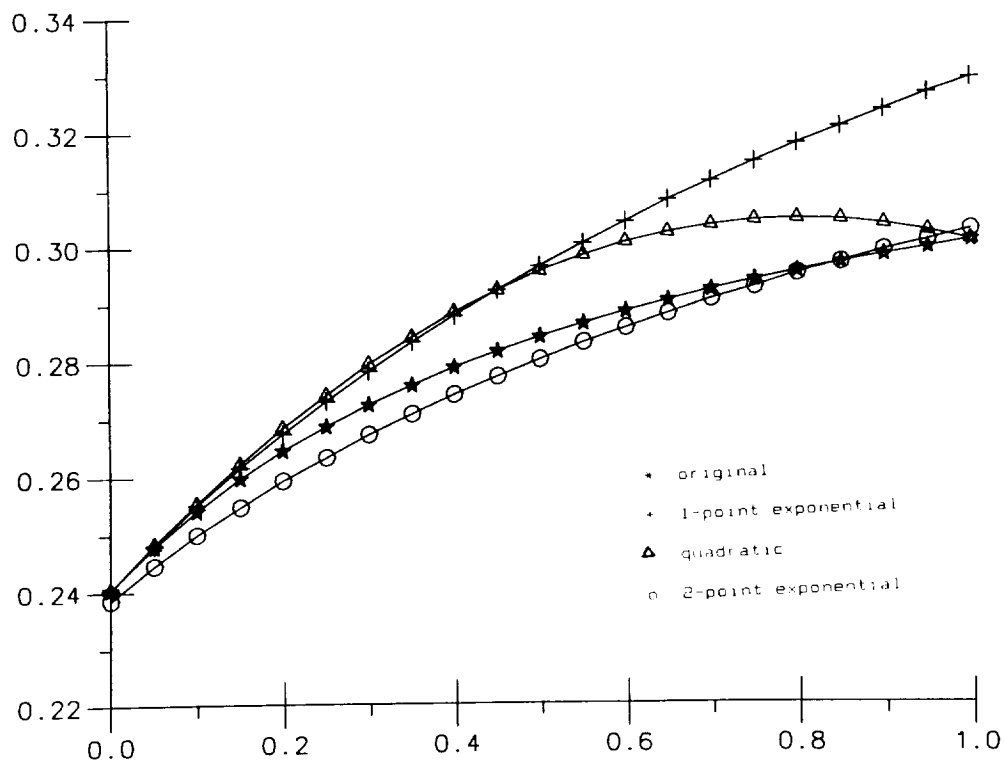


Figure 3. Approximations for the Three Bar Truss - Case I.

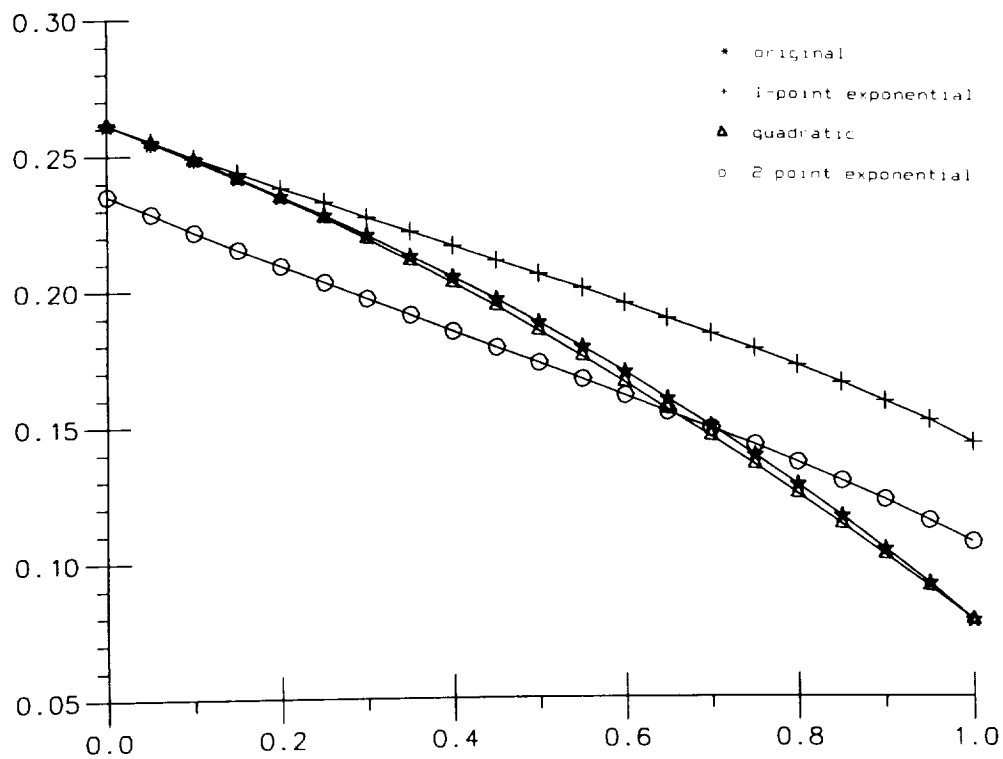


Figure 4. Approximations for the Three Bar Truss - Case II.

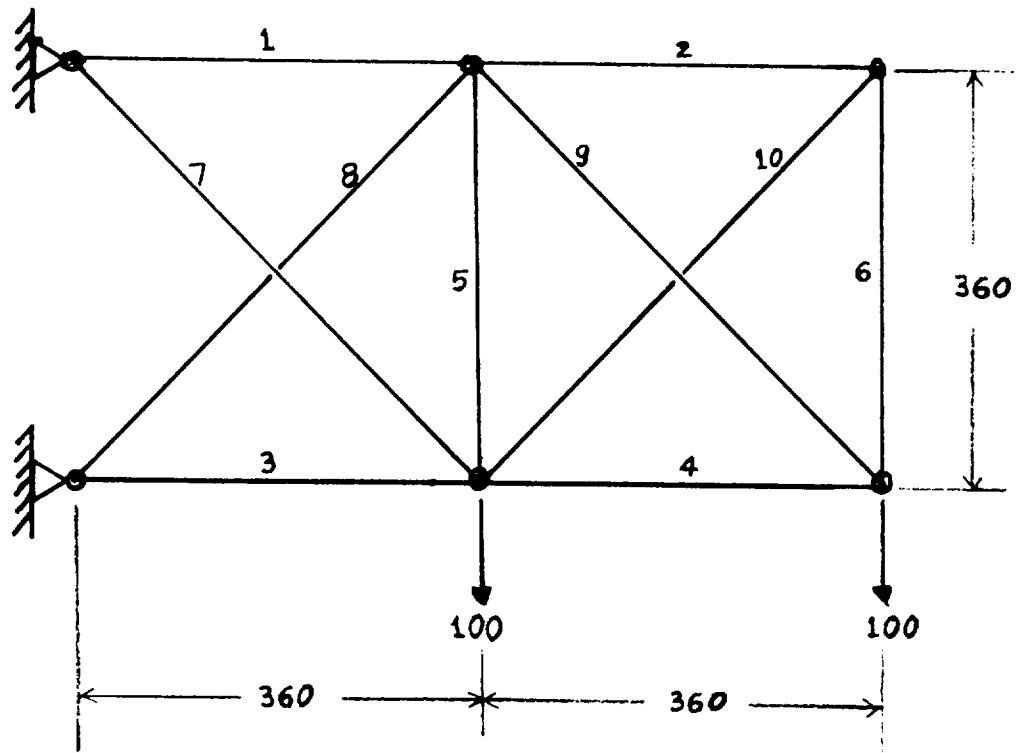


Figure 5. Ten Bar Truss Problem.

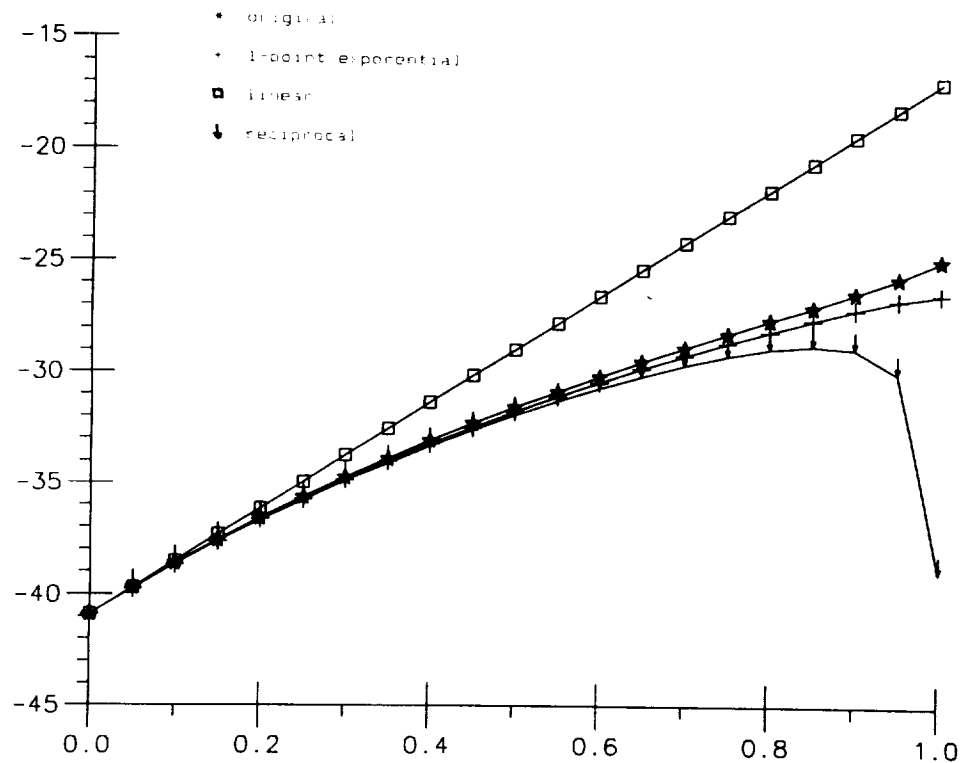


Figure 6. Approximations for the Ten Bar Truss.